

# A structural approach to pricing credit default swaps with credit and debt value adjustments

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## Abstract

A multi-dimensional extension of the structural default model with firms' values driven by diffusion processes with Marshall-Olkin-inspired correlation structure is presented. Semi-analytical methods for solving the forward calibration problem and backward pricing problem in three dimensions are developed. The model is used to analyze bilateral counterparty risk for credit default swaps and evaluate the corresponding credit and debt value adjustments.

## 1 Introduction

The recent financial crisis has profoundly changed the nature of credit markets in general and correlation trading in particular. The focus has shifted from complicated products, such as bespoke collateralized debt obligations (CDOs), CDOs-Squared, etc., towards simpler products, such as credit indices, collateralized credit default swaps (CDSs), funded single name credit-linked notes (CLNs), CDSs collateralized by a risky bond, etc, for which risks are somewhat easier to understand and model. However, as recent events have shown, if not properly managed, the trading of even these relatively simple products can cause big losses. More details can be found in Lipton and Rennie [2011].

During the crisis, it has become apparent that proper accounting for counterparty risk in the valuation of over-the-counter (OTC) derivatives is extremely important, especially in view of the fact that some protection sellers, such as mono-line insurers and investment banks, have experienced sharply elevated default probabilities or even default events, the case of Lehman Brothers being the prime example. Counterparty credit risk is the risk that a party to a financial contract will default prior to its expiration and will not fulfill all of its obligations. In principle, only OTC contracts privately negotiated between counterparties are subject to counterparty risk.

The structural model first introduced by Merton is one of the two standard models used for pricing single-name CDSs, the other one being the reduced-form model. Extensions of the structural model to the two-dimensional case have been proposed by Zhou [2001], Patras [2006], Valuzis [2008], among others, who considered correlated log-normal dynamics for two firms and derived analytical formulas for their joint survival probability using the eigenvalue expansion technique; see also Lipton [2001], He et al. [1998], where an identical technique was used in a different context. Two-dimensional structural models have been successfully used for the estimation of the credit value adjustment (CVA), and the debt value adjustment (DVA) for CDSs (see, e.g., Lipton and Sepp [2009], Blanchet-Scaillet and Patras [2011]).

In order to compute the CVA (DVA), one needs to study the joint evolution of the assets of the reference name and the protection seller (buyer), provided that the corresponding CDS is viewed from the standpoint of the protection buyer. Clearly, in order to calculate the CVA and DVA for a CDS *simultaneously and consistently*, one needs to consider three-dimensional structural models and study the joint evolution of the assets of the reference name, the protection seller and the protection buyer. This paper extends the results of Lipton and Sepp [2009] by considering correlated log-normal dynamics for three firms and computing their joint survival probability. The corresponding problem is solved by using the eigenfunction expansion technique combined with the finite element method to obtain a semi-analytical expression for the Green's function. Once the Green's function is known, both CVA and DVA corrections for a CDS can be computed in a consistent manner. The power of the proposed technique is illustrated by considering some realistic examples of pricing CDSs sold by risky sellers to risky buyers. As might be expected, counterparty credit effects have great impact on the value of a CDS contract.

## 2 CVA/DVA for CDSs

A CDS is a contract in which the protection buyer (PB) agrees to pay a periodic coupon  $c$  to a protection seller (PS) in exchange for a potential cashflow in the event of a default of the reference name (RN) of the swap before the maturity of the contract  $T$ . The value can be naturally decomposed into a coupon leg (CL) and a default leg (DL). We denote by  $\tau^{RN}$  the default time of the reference name and by  $R_{RN}$  its recovery, and we have (from the protection buyer's point of view):

$$\begin{aligned} CL_t &= -\mathbb{E} \left[ \sum_{T_i} cD(t, T_i) \mathbb{1}_{\{T_i \leq \tau^{RN}\}} \Delta T \mid \mathcal{F}_t \right], \\ DL_t &= \mathbb{E} \left[ (1 - R_{RN}) D(t, \tau^{RN}) \mathbb{1}_{\{t < \tau^{RN} < T\}} \mid \mathcal{F}_t \right], \end{aligned}$$

where  $T_i$  are the coupon payment dates and  $D(t, T)$  is the price of a zero-coupon bond with maturity  $T$ .

In order to simplify the formulas we denote by  $CF(t, T)$  the sum of all discounted contractual cashflows between  $t$  and the maturity  $T$  (both coupon leg and default leg), and write the value  $V_t$  of the CDS as:  $V_t = \mathbb{E}[CF(t, T) \mid \mathcal{F}_t]$ .

We suppose now that the protection seller can default but consider the protection buyer risk free, and denote by  $\tilde{V}_t$  the value of the derivative in this case:

$$\begin{aligned}\tilde{V}_t = & \mathbb{E} \left[ CF(t, T) \mathbb{1}_{\{\tau^{PS} > \min\{T, \tau^{RN}\}\}} \mid \mathcal{F}_t \right] \\ & + \mathbb{E} \left[ \mathbb{1}_{\{\tau^{PS} < \min\{T, \tau^{RN}\}\}} \left[ CF(t, \tau^{PS}) + D(t, \tau^{PS}) (R_{PS} V_{\tau^{PS}}^+ + V_{\tau^{PS}}^-) \right] \mid \mathcal{F}_t \right],\end{aligned}$$

where  $\tau^{PS}$  denotes the default time of the protection seller; as usual,  $V^\pm = \pm \max(0, \pm V)$ . The term Credit Value Adjustment (CVA) will refer to the additional cost to account for the possibility of the counterparty's default and is defined as  $CVA = V_t - \tilde{V}_t$ :

$$CVA = (1 - R_{PS}) \mathbb{E} \left[ \mathbb{1}_{\{\tau^{PS} < \min\{T, \tau^{RN}\}\}} D(t, \tau^{PS}) V_{\tau^{PS}}^+ \mid \mathcal{F}_t \right]. \quad (1)$$

Similarly we can consider the case where the protection buyer is risky but the protection seller is risk free. The term Debt Valuation Adjustment (DVA) represents the additional cost to account for one's own default ( $\tau^{PB}$  denotes the default time of the protection buyer):

$$DVA = (1 - R_{PB}) \mathbb{E} \left[ \mathbb{1}_{\{\tau^{PB} < \min\{T, \tau^{RN}\}\}} D(t, \tau^{PB}) V_{\tau^{PB}}^- \mid \mathcal{F}_t \right]. \quad (2)$$

Given recent events, we can no longer suppose that one of the counterparties is risk free. The Basel II documentation makes reference to a bilateral counterparty risk, in which the default of both counterparties in the derivative contract are subject to default risk. One of the advantages of considering the bilateral CVA is the symmetry it introduces in pricing: the two counterparties will now agree on the price of the derivative (for a detailed discussion on this see for example Brigo and Capponi [2010]). If  $\tau$  denotes the minimum of the two default times:  $\tau = \min\{\tau^{PS}, \tau^{PB}\}$ , then

$$\begin{aligned}\tilde{V}_t = & \mathbb{E} \left[ CF(t, T) \mathbb{1}_{\{\tau > T\}} + \right. \\ & + \mathbb{1}_{\{\tau = \tau^{PS} < T\}} \left( CF(t, \tau^{PS}) + D(t, \tau^{PS}) R_{PS} V_{\tau^{PS}}^+ + D(t, \tau^{PS}) V_{\tau^{PS}}^- \right) + \\ & \left. + \mathbb{1}_{\{\tau = \tau^{PB} < T\}} \left( CF(t, \tau^{PB}) + D(t, \tau^{PB}) R_{PB} V_{\tau^{PB}}^- + D(t, \tau^{PB}) V_{\tau^{PB}}^+ \right) \right].\end{aligned}$$

In the case where both counterparties are considered risky, bilateral CVA is the combination of the two adjustments (CVA and DVA):

$$CVA = (1 - R_{PS}) \mathbb{E} \left[ \mathbb{1}_{\{\tau^{PS} < \min\{T, \tau^{PB}, \tau^{RN}\}\}} D(t, \tau^{PS}) V_{\tau^{PS}}^+ \mid \mathcal{F}_t \right], \quad (3)$$

$$DVA = (1 - R_{PB}) \mathbb{E} \left[ \mathbb{1}_{\{\tau^{PB} < \min\{T, \tau^{PS}, \tau^{RN}\}\}} D(t, \tau^{PB}) V_{\tau^{PB}}^- \mid \mathcal{F}_t \right]. \quad (4)$$

We emphasize that expressions (1), (2) and (3), (4) are not identical.

### 3 Structural model framework

We assume that the default and counterparty risk can be hedged, so that we can work with the risk neutral pricing measure denoted by  $\mathbb{Q}$ . We also assume

a risk-free deterministic rate of return  $\varrho_t$ . We start with the firm's asset value dynamics, which we denote by  $a_t$ , and assume (similar to the setup in Lipton and Sepp [2009]) that it is driven by the following jump-diffusion dynamics under  $\mathbb{Q}$ :

$$\frac{da_t}{a_t} = (\varrho_t - \zeta_t - \lambda_t \kappa) dt + \sigma_t dW_t + (e^j - 1) dN_t, \quad (5)$$

where  $\zeta_t$  is the dividend rate,  $W_t$  is a standard Brownian motion,  $\sigma_t$  is the deterministic volatility,  $N_t$  is a Poisson process independent of  $W_t$ ,  $\lambda_t$  its intensity,  $j$  is the jump amplitude and  $\kappa$  is the jump compensator. We assume that the firm defaults when its asset value becomes less than a fraction of its debt per share and that the default barrier of the firm is a deterministic function of time given by  $l_t = l_0 E_t$ , where

$$E_t = \exp \left( \int_0^t (\varrho_u - \zeta_u - \lambda_u \kappa - \frac{1}{2} \sigma_u^2) du \right),$$

and  $l_0 = RL_0$ . Here  $R$  is the recovery on the firm's liabilities and  $L_0$  is its total debt per share. We consider the firm's equity price per share  $s_t$  and we assume that it is given by  $s_t = a_t - l_t$ , for  $t < \tau$  and 0 for  $t \geq \tau$  where  $\tau$  is the default time. The solution of the stochastic differential equation (5) can be written as a product of a deterministic part and a stochastic exponent  $a_t = l_0 E_t e^{\sigma_t x_t}$ , where the stochastic factor  $x_t$  has the following dynamics under  $\mathbb{Q}$ :

$$dx_t = dW_t + \frac{j}{\sigma_t} dN_t, \quad x_0 > 0, \quad (6)$$

with  $x_t$  representing the “relative distance” of the asset value from the default barrier. The default event occurs at the first time  $\tau$  when  $x_\tau$  becomes negative, so that the default barrier is fixed at zero.

Introducing jumps in the dynamics of the asset value allows us to calibrate to CDS market spreads even for short maturities. In a framework without jumps it is well known that a good calibration of the short end of the curve is impossible. However, this simpler framework allows for analytical solutions in some cases which provide insight into the problem as well as a good benchmark for the more general case with jumps. In this paper we therefore focus on the simplified case without jumps.

For the multi-dimensional case we consider that the process for the relative distance to default for each of the entities of interest has a similar dynamics to equation (6) but also in the simplified framework without the jump component, and we correlate the diffusions in the usual way by assuming  $d\langle W_t^i, W_t^j \rangle = \rho_{ij} dt$ .

## 4 Two dimensional case

The one dimensional case of the standard CDS where both counterparties are non-risky admits well-known analytical solutions. We therefore turn our focus directly to the two dimensional problem where we need to model the dynamics of the asset values of the reference name and protection seller simultaneously, while

considering the protection buyer to be non-risky. We work with the processes  $x_t$  and  $y_t$  for the relative distance from the default barrier in time for each of the two entities considered. These processes have the following dynamics  $dx_t = dW_t^x$ ,  $dy_t = dW_t^y$ , where the Brownian motions are correlated with correlation  $\rho_{xy}$ .

#### 4.1 Pricing problem

The general pricing equation in this framework is given by:

$$V_t + \frac{1}{2}V_{xx} + \frac{1}{2}V_{yy} + \rho_{xy}V_{xy} - \varrho V = 0. \quad (7)$$

We consider the following function  $U(t, x, y) = e^{\varrho(T-t)}V(t, x, y)$  and make a change of variable that allows us to eliminate the cross derivatives:

$$\alpha(x, y) = x, \quad \beta(x, y) = -\frac{1}{\bar{\rho}_{xy}}(\rho_{xy}x - y), \quad (8)$$

where we have used the usual notation  $\bar{\rho}_{xy} = \sqrt{1 - \rho_{xy}^2}$ . The domain in which the equation has to be solved has changed from the positive quadrant to the interior of an angle. This angle is characterized by  $\cos(\varphi_0) = -\rho_{xy}$ , so if  $\rho_{xy} > 0$ , the angle is blunt. In order to take advantage of the symmetry of the domain, we make a second change of variables and pass to polar coordinates,  $(\alpha, \beta) = (-r \sin(\varphi - \varphi_0), r \cos(\varphi - \varphi_0))$ .

#### 4.2 Green's function

The Green's function solves the forward problem (where  $\tau = T - t$ ):

$$G_\tau - \frac{1}{2} \left( G_{rr} + \frac{1}{r} G_r + \frac{1}{r^2} G_{\varphi\varphi} \right) = 0, \quad (9)$$

$$G(0, r, \varphi) = \frac{1}{r'} \delta(r - r') \delta(\varphi - \varphi')$$

$$G(\tau, r, 0) = 0 \quad G(\tau, r, \varphi_0) = 0 \quad G(\tau, 0, \varphi) = 0 \quad G(\tau, r, \varphi) \xrightarrow{r \rightarrow \infty} 0.$$

Two possible methods can be applied in order to obtain the solution for the Green's function: the eigenvalue expansion method and the method of images. The solution using the first method is well known and was first introduced in Zhou [2001], Lipton [2001], He et al. [1998]. The resulting formula for the Green's function is:

$$G(\tau, r, \varphi | 0, r', \varphi') = \frac{2e^{-\frac{r^2 + r'^2}{2\tau}}}{\varphi_0 \tau} \sum_{n=1}^{\infty} I_{\frac{n\pi}{\varphi_0}} \left( \frac{rr'}{\tau} \right) \sin \left( \frac{n\pi\varphi}{\varphi_0} \right) \sin \left( \frac{n\pi\varphi'}{\varphi_0} \right). \quad (10)$$

A solution through the method of images was announced by Lipton in 2008 at a SIAM meeting, and briefly discussed in Lipton and Sepp [2009]. Here we

present an improved version. We first need to find the solution to equation (9) with the same initial condition but with non-periodic boundary conditions:

$$G(\tau, 0, \varphi) = 0 \quad G(\tau, r, \varphi) \xrightarrow{r \rightarrow \infty} 0 \quad G(\tau, r, \varphi) \xrightarrow{|\varphi| \rightarrow \infty} 0.$$

Using a Fourier transform technique, and in order to write the final expression in a more compact form we introduce the following function  $f(p, q)$  where  $p \geq 0, -\infty < q < \infty$ :

$$f(p, q) = 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-p(\cosh(2q\zeta) - \cos(q))}}{\zeta^2 + \frac{1}{4}} d\zeta,$$

and its extension  $h(p, q)$  defined as follows:

$$h(p, q) = \frac{1}{2} [s_+ f(p, \pi + q) + s_- f(p, \pi - q)],$$

where  $s_{\pm} = \text{sign}(\pi \pm q)$ . Then we can represent  $G(t, r, \varphi|0, r', \varphi')$  in the following form (which can be viewed as a direct generalization of the one dimensional case):

$$G(\tau, r, \varphi|0, r', \varphi') = \frac{1}{2\pi\tau} e^{-\frac{r^2 + r'^2 - 2\cos(\varphi - \varphi')rr'}{2\tau}} h\left(\frac{rr'}{\tau}, \varphi - \varphi'\right).$$

We can now apply the method of images and represent the fundamental solution in the form

$$G_{\varphi_0}(\tau, r, \varphi|0, r', \varphi') = \sum_{n=-\infty}^{\infty} [G(\tau, r, \varphi|0, r', \varphi' + 2n\varphi_0) - G(\tau, r, \varphi|0, r', -\varphi' + 2n\varphi_0)]. \quad (11)$$

The representation given in equation (11) gives, as expected, exactly the same results for the Green's function as those obtained through the eigenvalue expansion method.

### 4.3 Joint survival probability

We denote by  $Q(t, x, y)$  the joint survival probability of issuers  $x$  and  $y$  to a fixed maturity  $T$ . This solves the following pricing equation

$$Q_t + \frac{1}{2}Q_{xx} + \frac{1}{2}Q_{yy} + \rho_{xy}Q_{xy} = 0,$$

with final condition  $Q(T, x, y) = 1$  and boundary conditions  $Q(t, x, 0) = 0$  and  $Q(t, 0, y) = 0$ . We use the expression for the Green's function obtained through the eigenvalue expansion method and we obtain the following expression for the survival probability in the new variables:

$$Q(t, r', \varphi') = \frac{4}{\pi} \int_0^{\infty} \frac{e^{-\frac{r^2 + r'^2}{2\tau}}}{\tau} \sum_{k=0}^{\infty} \frac{1}{2k+1} I_{\frac{(2k+1)\pi}{\varphi_0}}\left(\frac{rr'}{\tau}\right) \sin \frac{(2k+1)\pi\varphi'}{\varphi_0} r dr.$$

#### 4.4 Application to the CVA computation

We associate the process  $x_t$  with the protection seller and the process  $y_t$  with the reference name issuer of a CDS. The protection buyer will be considered non-risky in this case. The pricing equation for computing the CVA is given by:

$$V_t + \frac{1}{2}V_{xx} + \frac{1}{2}V_{yy} + \rho_{xy}V_{xy} - \varrho V = 0, \quad (12)$$

with the final condition  $V(T, x, y) = 0$ . Boundary conditions are 0, except in the case when the protection seller defaults first and there will be a shortfall equal to a fraction of the outstanding present value of the standard single name swap:

$$V^{\text{CVA}}(t, 0, y) = (1 - R_{PS}) V^{\text{CDS}}(t, y)^+.$$

where  $R_{PS}$  is the recovery of the protection seller. In order to solve this problem we apply similar changes of function and variables as in section 4.1 and obtain a similar pricing equation

$$U_t + \frac{1}{2} \left( U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\varphi\varphi} \right) = 0, \quad (13)$$

with the final condition:  $U(T, r, \varphi) = 0$  and boundary conditions:

$$\begin{aligned} U(t, 0, \varphi) &= 0, \quad U(t, \infty, \varphi) = 0, \quad U(t, r, 0) = 0, \\ U^{\text{CVA}}(t, r, \varphi_0) &= e^{\varrho(T-t)} (1 - R_{PS}) V^{\text{CDS}}(t, \bar{\rho}_{xy}r)^+. \end{aligned}$$

The solution for this problem is given by:

$$\begin{aligned} U(t, r', \varphi') &= \frac{1}{2} \left[ \int_t^T \int_0^\infty \frac{\partial G(t' - t, r, \varphi)}{\partial \varphi} \Big|_{\varphi=0} U(t', r, 0) \frac{1}{r} dr dt' \right. \\ &\quad \left. - \int_t^T \int_0^\infty \frac{\partial G(t' - t, r, \varphi)}{\partial \varphi} \Big|_{\varphi=\varphi_0} U(t', r, \varphi_0) \frac{1}{r} dr dt' \right]. \end{aligned}$$

Supplying the boundary conditions for the CVA problem we obtain:

$$V^{\text{CVA}}(t, r', \varphi') = -\frac{1-R_{PS}}{2} \int_t^T \int_0^\infty \frac{\partial G(t' - t, r, \varphi)}{\partial \varphi} \Big|_{\varphi=\varphi_0} e^{-\varrho(t'-t)} V^{\text{CDS}}(t', \bar{\rho}_{xy}r)^+ \frac{1}{r} dr dt'. \quad (14)$$

Thus, by using Green's function, we can perform the CVA computation in a natural and straightforward way.

### 5 Three dimensional case

For the three dimensional problem we need to model the dynamics of the asset values of the reference name, protection seller and protection buyer simultaneously. As shown previously, we consider the default barrier to be a deterministic

function of time for all three assets and we work directly with the processes  $x_t$ ,  $y_t$  and  $z_t$  which measure the “relative” distance from the default barrier in time for each of the three entities considered. These processes have the following dynamics:  $dx_t = dW_t^x$ ,  $dy_t = dW_t^y$ ,  $dz_t = dW_t^z$ , where we correlate the Brownian motions with correlations  $\rho_{xy}$ ,  $\rho_{xz}$ ,  $\rho_{yz}$ .

## 5.1 Pricing problem

The general pricing problem in the  $\mathbb{R}_+^3$  octant is:

$$V_t + \frac{1}{2}V_{xx} + \frac{1}{2}V_{yy} + \frac{1}{2}V_{zz} + \rho_{xy}V_{xy} + \rho_{xz}V_{xz} + \rho_{yz}V_{yz} - \rho V = 0.$$

We consider the following function  $U(t, x, y, z) = e^{\rho(T-t)}V(t, x, y)$ , and introduce a change of variables that allows us to eliminate the cross derivatives:

$$\alpha = x, \quad \beta = \frac{(-\rho_{xy}x + y)}{\bar{\rho}_{xy}}, \quad \gamma = \frac{((\rho_{xy}\rho_{yz} - \rho_{xz})x + (\rho_{xy}\rho_{xz} - \rho_{yz})y + \bar{\rho}_{xy}z)}{\bar{\rho}_{xy}\chi}, \quad (15)$$

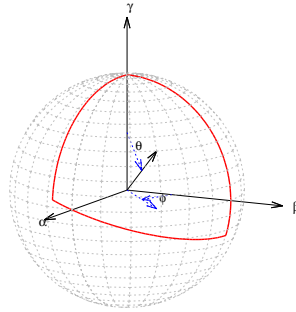
where we use the notation:  $\chi = \sqrt{1 - \rho_{xy}^2 - \rho_{xz}^2 - \rho_{yz}^2 + 2\rho_{xy}\rho_{xz}\rho_{yz}}$ .

With the change of variables, we have also changed the domain in which we need to solve the pricing problem. The domain becomes the volume bounded by the planes:  $\alpha = 0$ ,  $(\alpha, -\frac{\rho_{xy}}{\bar{\rho}_{xy}}\alpha, \gamma)$  and  $(\alpha, \beta, \frac{\bar{\rho}_{xy}}{\chi}(-\rho_{xz}\alpha + \frac{\rho_{xy}\rho_{xz} - \rho_{yz}}{\bar{\rho}_{xy}}\beta))$ . In order to take advantage of the symmetry of the problem we perform a second change of variables to spherical coordinates: the axis  $\alpha = 0$  and  $\beta = 0$  is given by  $\theta = 0$ ; the axis  $\alpha = 0$  and  $\gamma = 0$  is given by  $\varphi = 0$  and  $\theta = \pi/2$ , so that  $(\alpha, \beta, \gamma) = (r \sin \theta \sin \varphi, r \sin \theta \cos \varphi, r \cos \theta)$ .

The range of values for  $\varphi$  is given by:  $0 \leq \varphi \leq \varphi_0$  where  $\varphi_0 = \arccos(-\rho_{xy})$ . As can be observed in figure 1, the possible range of values for  $\theta$  will depend on  $\varphi$ , so we have  $0 \leq \theta \leq \Theta(\varphi)$ . Formulas (16) and (17) give a parametric characterization of this boundary of the domain which will prove very useful going forward.

$$\varphi(\omega) = \arccos\left(\frac{1 - \rho_{xy}\omega}{\sqrt{1 - 2\rho_{xy}\omega + \omega^2}}\right), \quad (16)$$

Figure 1: Domain after the change in coordinates for  $\rho_{xy} = 20\%$ ,  $\rho_{xz} = 0\%$ ,  $\rho_{yz} = 30\%$





$$\Theta(\omega) = \arccos \left( -\frac{\rho_{yz} - \rho_{xz}\rho_{xy} + \omega(\rho_{xz} - \rho_{yz}\rho_{xy})}{\sqrt{\rho_{xy}(\bar{\rho}_{xz}^2 - 2\omega(\rho_{xy} - \rho_{xz}\rho_{yz}) + \omega^2\bar{\rho}_{yz}^2)}} \right). \quad (17)$$

In the domain described above the final form of the pricing equation is:

$$U_t + \frac{1}{2} \left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} (rU) + \frac{1}{r^2} \left( \frac{1}{\sin^2 \theta} U_{\varphi\varphi} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta U_\theta) \right) \right] = 0, \quad (18)$$

with appropriate boundary conditions depending on the payoff.

## 5.2 Green's function

We now concentrate on solving the forward problem for the Green's function in spherical coordinates:

$$\begin{aligned} G_\tau - \frac{1}{2} \left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} (rG) + \frac{1}{r^2} \left( \frac{1}{\sin^2 \theta} G_{\varphi\varphi} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta G_\theta) \right) \right] &= 0, \quad (19) \\ G(0, r, \varphi, \theta) &= \frac{1}{r^2 \sin \theta} \delta(r - r') \delta(\varphi - \varphi') \delta(\theta - \theta'), \\ G(\tau, r, 0, \theta) &= G(\tau, r, \varphi_0, \theta) = G(\tau, r, \varphi, 0) = 0, \\ G(\tau, r, \varphi, \Theta(\varphi)) &= G(\tau, 0, \varphi, \theta) = 0, \quad G(\tau, r, \varphi, \theta) \xrightarrow{r \rightarrow \infty} 0. \end{aligned}$$

We aim to build a solution for the Green's function through the eigenvalues expansion method. The first step is to apply the separation of variables technique:

$$G(\tau, r, \varphi, \theta) = g(\tau, r) \Psi(\varphi, \theta). \quad (20)$$

By substituting (20) in (19) we obtain an equation where the left hand side depends only on  $\tau$  and  $r$  and the right hand side depends only on  $\varphi$  and  $\theta$  and hence both sides are equal to some constant value  $C$ , which is necessarily negative; we use the notation  $C = -\Lambda^2$ .

For function  $g(\tau, r)$  we have the initial condition  $g(0, r) = \frac{1}{r^2} \delta(r - r')$ , and boundary conditions  $g(\tau, 0) = 0$  and  $g(\tau, r) \xrightarrow{r \rightarrow \infty} 0$ . Function  $g$  solves the following PDE

$$g_\tau = \frac{1}{2} \left( \frac{1}{r} \frac{\partial^2}{\partial r^2} (rg) - \frac{\Lambda^2}{r^2} g \right),$$

which is similar to the equation utilised for the two dimensional case. The solution is given by:

$$g(\tau, r) = \frac{e^{-\frac{r^2 + r'^2}{2\tau}}}{\tau \sqrt{rr'}} I_{\sqrt{\Lambda^2 + 1/4}} \left( \frac{rr'}{\tau} \right).$$

In order to obtain the Green's function we also need to solve the angular part PDE:

$$\frac{1}{\sin^2 \theta} \Psi_{\varphi\varphi} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \Psi_\theta) = -\Lambda^2 \Psi, \quad (21)$$

with zero boundary conditions:  $\Psi(0, \theta) = \Psi(\varphi_0, \theta) = \Psi(\varphi, 0) = \Psi(\varphi, \Theta(\varphi)) = 0$ . It is well-known that the spectrum of this problem is discrete and the set of the corresponding eigenvectors is complete.

The two dimensional spherical surface inside the red line shown in figure 1 may be mapped directly onto the  $(\varphi, \theta)$  plane. This is done in a similar way to the method used by cartographers to map the Earth's surface using Mercator's projection. The southern boundary of the domain is mapped into a continuous curve parametrised by equations (16) and (17). The boundary at  $\theta = 0$  is degenerate as it corresponds to the north pole on the sphere. Figure 2 and 3 show the domain (denoted by  $\Omega$ ) projected onto the  $(\varphi, \theta)$  plan for sample correlation values.

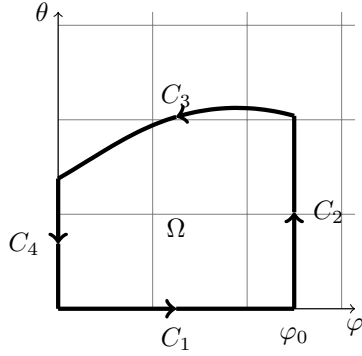


Figure 2:  $\rho_{xy} = 0.8$ ,  $\rho_{xz} = 0.5$ ,  $\rho_{yz} = 0.3$

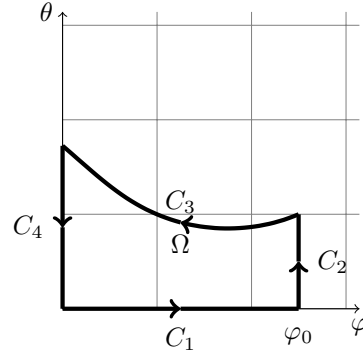


Figure 3:  $\rho_{xy} = 0.8$ ,  $\rho_{xz} = -0.65$ ,  $\rho_{yz} = -0.45$

In our case we are interested in writing an eigenvalue expansion for the Green's function. The weak formulation for our problem is given by

$$\int_{\Omega} \frac{1}{\sin \theta} \Psi_{\varphi} \Psi'_{\varphi} d\Omega + \int_{\Omega} \sin \theta \Psi_{\theta} \Psi'_{\theta} d\Omega = \Lambda^2 \int_{\Omega} \Psi \Psi' \sin \theta d\Omega, \quad (22)$$

where  $\Psi'$  is a test function that belongs to the same space as  $\Psi$ , in particular it is also 0 on the boundary of the domain.

The first step necessary is to construct a mesh on the domain of interest. We construct a triangular mesh following the ideas presented in Persson [2005]. For the actual mesh generation, the algorithm uses an iterative technique. We use adaptive grids that are finer along the boundaries.

Figure 4 shows an example of mesh obtained for sample values of the correlations. We show the initial mesh and the final mesh obtained after 100 iterations.

Once the mesh is constructed we solve the eigenvalue problem in matrix form and obtain the eigenvalues and corresponding eigenvectors. Figure 5 shows sample eigenvectors for a domain where all three correlations are positive.

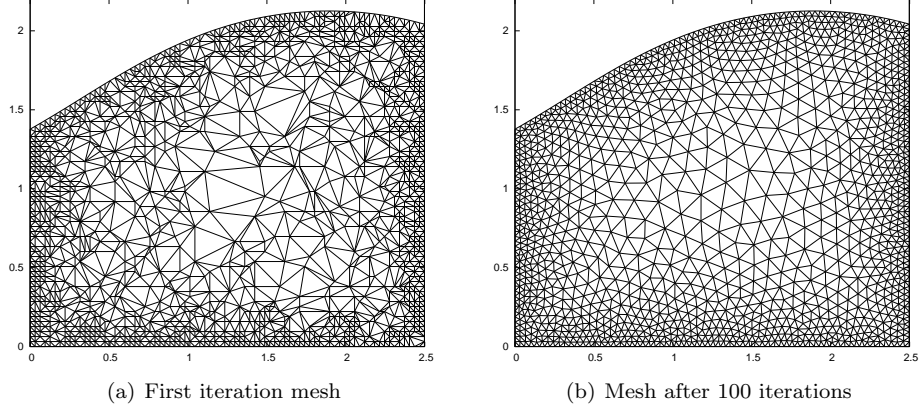


Figure 4: Adaptive mesh for the domain obtained for  $\rho_{xy} = 80\%$ ,  $\rho_{xz} = 50\%$ ,  $\rho_{yz} = 30\%$ . The mesh is constructed using 1500 points and is finer as we get closer to the boundaries.

Having calculated the eigenvectors and eigenvalues for our problem we can write the eigenfunction expansion for  $\Psi(\varphi, \theta)$ , and then for the Green's function we obtain the following final formula:

$$G(\tau, r, \varphi, \theta | r', \varphi', \theta') = \frac{e^{-\frac{r^2 + r'^2}{2\tau}}}{\tau \sqrt{rr'}} \sum_{n=1}^{\infty} I_{\sqrt{\Lambda_n^2 + \frac{1}{4}}} \left( \frac{rr'}{\tau} \right) \Psi_n(\varphi', \theta') \Psi_n(\varphi, \theta). \quad (23)$$

### 5.3 Joint survival probability

Similarly to the two dimensional case, we denote by  $Q(t, x, y, z)$  the joint survival probability of issuers  $x, y$  and  $z$  to a fixed maturity  $T$ . This solves the following pricing equation

$$Q_t + \frac{1}{2}Q_{xx} + \frac{1}{2}Q_{yy} + \frac{1}{2}Q_{zz} + \rho_{xy}Q_{xy} + \rho_{xz}Q_{xz} + \rho_{yz}Q_{yz} = 0 \quad (24)$$

with final condition  $Q(T, x, y, z) = 1$  and zero boundary conditions. We proceed with a similar change of variables as described in section 5.1, and using the expression for the Green's function given in equation (23) we obtain:

$$Q(t, r', \varphi', \theta') = \int_0^\infty \frac{e^{-\frac{r^2 + r'^2}{2\tau}}}{\tau \sqrt{rr'}} \sum_{n=1}^{\infty} I_{\sqrt{\Lambda_n^2 + \frac{1}{4}}} \left( \frac{rr'}{\tau} \right) \Psi_n(\varphi', \theta') \left[ \iint_{\Omega} \Psi_n(\varphi, \theta) \sin \theta d\Omega \right] r^{\frac{3}{2}} dr.$$

### 5.4 Application to the CVA computation

We associate the process  $x_t$  with the protection seller, the process  $y_t$  with the reference name and  $z_t$  with the protection buyer. The pricing equation for

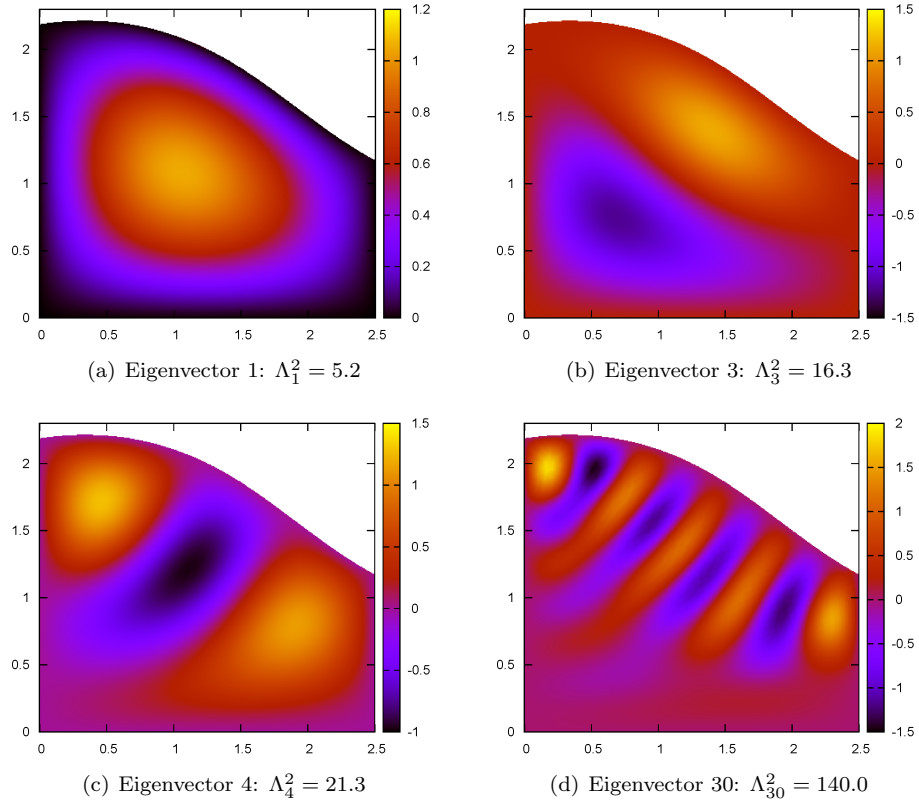


Figure 5: Eigenvectors and corresponding eigenvalues for the domain obtained for  $\rho_{xy} = 80\%$ ,  $\rho_{xz} = 20\%$ ,  $\rho_{yz} = 50\%$ .

computing CVA or DVA in the case where all three names are risky is given by:

$$V_t + \frac{1}{2}V_{xx} + \frac{1}{2}V_{yy} + \frac{1}{2}V_{zz} + \rho_{xy}V_{xy} + \rho_{xz}V_{xz} + \rho_{yz}V_{yz} - \varrho V = 0, \quad (25)$$

with the final condition  $V(T, x, y, z) = 0$  and boundary conditions depending on the payoff.

In the case of the CVA calculation, we get a payout if the protection seller defaults, the payout is:

$$V^{\text{CVA}}(t, 0, y, z) = (1 - R_{PS}) V(t, y)^+, \quad (26)$$

where  $V(t, y)^+$  is the positive value of the single name default swap with non-risky counterparts at the time of the default of the protection seller.

Similarly we have the payout for the DVA calculation:

$$V^{\text{DVA}}(t, x, y, 0) = (1 - R_{PB}) V(t, y)^-, \quad (27)$$

where  $R_{PB}$  is the recovery of the protection buyer and  $V(t, y)^-$  is the negative value of the single name default swap with non-risky counterparts at the time of the default of the protection buyer. For both CVA and DVA calculations the boundary conditions are 0 for all other cases. Following the same procedure as in section 5.1 we obtain the modified pricing equation

$$U_t + \frac{1}{2} \left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} (rU) + \frac{1}{r^2} \left( \frac{1}{\sin^2 \theta} U_{\varphi\varphi} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta U_\theta) \right) \right] = 0, \quad (28)$$

with final condition  $U(T, r, \varphi, \theta) = 0$  and 0 boundary conditions except for:

$$U^{\text{CVA}}(t, r, 0, \theta) = e^{\varrho(T-t)} (1 - R_{PS}) V(t, \bar{\rho}_{xy} r \sin \theta)^+, \quad (29)$$

$$U^{\text{DVA}}(t, r, \varphi, \theta(\varphi)) = e^{\varrho(T-t)} (1 - R_{PB}) V(t, r \sin \theta (\rho_{xy} \sin \varphi + \bar{\rho}_{xy} \cos \varphi))^- \quad (30)$$

for the CVA and DVA respectively.

The final pricing formula for  $U$  is given by:

$$\begin{aligned} U(t, r, \varphi, \theta) = & -\frac{1}{2} \int_t^T \int_0^\infty \int_0^{\varphi_0} \sin \theta(\varphi) U(t', r, \varphi, \theta(\varphi)) G_\theta(t', r, \varphi, \theta(\varphi)) d\varphi dr dt' \\ & + \frac{1}{2} \int_t^T \int_0^\infty \int_0^\infty \frac{U(t', r, \varphi(\omega), \theta(\omega)) G_\varphi(t', r, \varphi(\omega), \theta(\omega))}{\sin \theta(\omega)} \theta'(\omega) d\omega dr dt' \\ & - \frac{1}{2} \int_t^T \int_0^\infty \int_0^{\theta(\varphi_0)} \frac{U(t', r, \varphi_0, \theta) G_\varphi(t', r, \varphi_0, \theta)}{\sin \theta} d\theta dr dt' \\ & + \frac{1}{2} \int_t^T \int_0^\infty \int_0^{\theta(0)} \frac{U(t', r, 0, \theta) G_\varphi(t', r, 0, \theta)}{\sin \theta} d\theta dr dt'. \end{aligned} \quad (31)$$

We note that for one of the integrals above we have used the parametric representation of the boundary of our domain given by formulas (16) and (17). To obtain the precise formulas for the CVA and DVA calculations we now apply the boundary conditions in equations (29) and (30) respectively:

$$U^{\text{CVA}}(t, r', \varphi', \theta') = \frac{1}{2} \int_t^T \int_0^\infty \int_0^{\theta(0)} \frac{U^{\text{CVA}}(t', r, 0, \theta) G_\varphi(t', r, 0, \theta)}{\sin \theta} d\theta dr dt', \quad (32)$$

$$\begin{aligned} U^{\text{DVA}}(t, r', \varphi', \theta') = & -\frac{1}{2} \int_t^T \int_0^\infty \int_0^{\varphi_0} \sin \theta(\varphi) U^{\text{DVA}}(t', r, \varphi, \theta(\varphi)) G_\theta(t', r, \varphi, \theta(\varphi)) d\varphi dr dt' \\ & + \frac{1}{2} \int_t^T \int_0^\infty \int_0^\infty \frac{U^{\text{DVA}}(t', r, \varphi(\omega), \theta(\omega)) G_\varphi(t', r, \varphi(\omega), \theta(\omega))}{\sin \theta(\omega)} \theta'(\omega) d\omega dr dt'. \end{aligned} \quad (33)$$

These original formulas provide a new way of consistently computing the CVA and DVA. Similar ideas can be used for many other purposes, which will be discussed elsewhere.

## 6 Numerical results

In this section we present the results of the CVA and DVA calculations for a risky CDS. We compare the breakeven coupon obtained for a standard CDS to those obtained when either the protection buyer or the protection seller are risky (using the 2D formulation and results), as well as when both are risky (using the 3D formulation and results). When using the 2D formulation and considering that either the protection seller or the protection buyer are risky, the two parties will not agree on the breakeven coupon of the CDS. This problem goes away when using the full three dimensional framework in which both are risky and the problem becomes symmetrical.

We consider three issuers for our example: AIG as a protection seller, GE as a reference name of the CDS, and UNICREDIT a protection buyer. We have chosen sufficiently risky entities for the protection seller and the protection buyer in order to emphasize the effect of the CVA and DVA adjustments on the break even coupon. We calibrate the model inputs to the market data from the 15th of December 2011 (see table 1).

The initial value is a measure of the relative distance to default. The volatility  $\sigma$  has been calibrated such that the 5Y single name CDS spread is matched to the market spread (the 5Y point has been chosen as it is usually the most liquidly traded contract).

For the 2D and 3D cases we also need the correlations between the different issuers as inputs to our model. These can be calibrated from the prices of the

first to default swap contracts if such contracts including the relevant names are available on the market. Alternatively we can proxy these correlations by assigning a sector to each issuer and then using the sector to sector historically estimated correlations. In this section however we aim to show the impact of CVA and DVA on the breakeven spread of a CDS, and hence we will use different sets of pairwise correlations for the same group of issuers in order to illustrate a variety of cases.

Figure 6 shows the case where the protection seller is highly correlated to the reference name. We observe that the spreads are hyper-exponentially flat at 0, which is a known problem for models without jumps. However for longer maturities we can match model and market prices. We use the calibrated model in order to analyse the effects of either the protection seller, or the protection buyer, or both being risky.

If the protection seller is risky, the probability of it not paying the full amount due in the case of the default of the reference name is non-zero, and hence the protection buyer pays a lower coupon as it takes on that risk as well. If the protection buyer is risky then the breakeven coupon moves in the opposite direction and the two counterparties no longer agree on the coupon. Typically, the latter shift is much smaller than the former. In the case when both are considered risky the problem becomes symmetrical, and a mutually agreeable coupon can be computed.

In this example, in the case of a default of the reference name, the protection seller is likely to default as well, and hence the shortfall between the contractual payout and what will actually be paid can be significant. The break-even coupon will adjust accordingly and will be lower than on a standard fully collateralised CDS as the expectation of the payout is lower from the protection buyer's point of view.

Figure 7 shows the case where the protection buyer is highly anti-correlated to the reference name. This is intuitively the case where the DVA is largest as it is in the cases where the reference name does not default that the protection buyer is more likely to default on its coupon paying obligation. This leaves the protection seller with a potential shortfall.

## 7 Conclusion

A 3D extension of the structural default framework where the joint dynamics of the firms' values are driven by correlated Brownian motions is presented. A

Inputs	AIG	GE	UNICREDIT
Initial value	0.0359	0.3035	0.1199
$\sigma$	2.44%	10.45%	6.3%
Recovery	50%	40%	40%

Table 1: Input parameters calibrated to market data (15th December 2011)

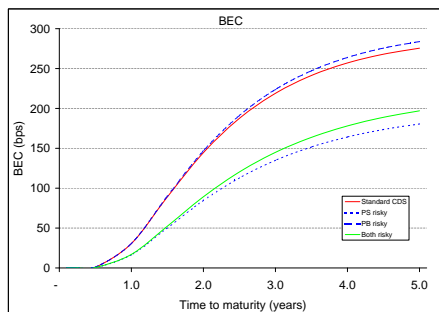


Figure 6:  $\rho_{xy} = 80\%$ ,  $\rho_{xz} = 50\%$ ,  $\rho_{yz} = 30\%$

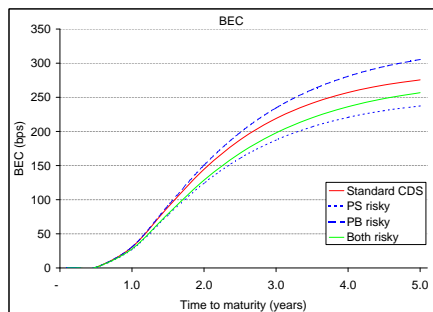


Figure 7:  $\rho_{xy} = 20\%$ ,  $\rho_{xz} = -10\%$ ,  $\rho_{yz} = -60\%$

method to obtain a semi-analytical expression for the Green's function using the eigenvalue expansion method is developed. It is shown how to apply this in order to compute joint survival probabilities of three different companies and how to calculate the credit and debit value adjustments for a standard CDS. In the 3D case, a fully analytical expression is not available, since the eigenvalues and eigenvectors have to be computed using the finite element method. Given a triplet of correlations, however, these can be precomputed, which then allows efficient computations across a range of initial points, volatilities or other trade-related data (coupons, recoveries etc). Concrete examples demonstrate that the CVA and DVA for a typical CDS can be very large.

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